INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2014-15
Statistics - III, Semesteral Examination, November 7, 2014
Marks are shown in square brackets. Total Marks: 50

1. For $n \geq 4$ let $Z_i, 1 \leq i \leq n$ be independent $N(0, \sigma^2)$ random variables. Consider $0 < \alpha < 1$. Define $X_1 = Z_1$ and $X_{i+1} = -\alpha X_i + \sqrt{1-\alpha^2} Z_{i+1}$ for $1 \leq i \leq n-1$. Let $\mathbf{X} = (X_1, \ldots, X_n)'$.
(a) Find the probability distribution of $\mathbf{X}$.
(b) Find the partial correlation coefficients $\rho_{12.3}$, $\rho_{13.2}$ and $\rho_{14.23}$ (between elements of $\mathbf{X}$). [11]

2. Consider the model $\mathbf{Y} = X\beta + \epsilon$, where $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$ and $X_{n \times p}$ has rank $r \leq p$ and its first column is $\mathbf{1}$.
(a) Define the coefficient of determination for this model. What does it measure?
(b) If $n = 14$, $p = 6$, $r = 4$ and $R^2 = 80\%$, compute the F-ratio for testing the usefulness of the regressors. [11]

3. Consider the model:

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij},$$

$1 \leq i \leq 4$, $1 \leq j \leq 10$, where ϵ_{ij} are i.i.d. $N(0, \sigma^2)$; also assume the usual constraints on the parameters for identifiability.
(a) Explain why constraints are needed for identifiability of parameters.
(b) Show that $\alpha_1 - \alpha_4$ is estimable, and find its BLUE.
(c) Provide a 95% confidence interval for $\alpha_1 - \alpha_4$.
(d) Find the maximum likelihood estimator of σ^2. Is it unbiased? [14]

4. Suppose $\mathbf{X} \sim N_p(\mathbf{0}, \mathbf{\Sigma})$ where $\text{Rank}(\Sigma) = r \leq p$ and let B and C be any symmetric matrices.
(a) Show that $\mathbf{X}'B\mathbf{X}$ and $\mathbf{X}'C\mathbf{X}$ are independent χ^2 random variables if and only if

$$\Sigma B \Sigma B \Sigma = \Sigma B \Sigma, \ \Sigma C \Sigma C \Sigma = \Sigma C \Sigma, \ \Sigma B \Sigma C \Sigma = \mathbf{0}.$$

(b) Find the degrees of freedom of these χ^2 distributions. [14]